Master in Actuarial Science

Risk Theory

27-06-2011

Solution

1. -

   (a) $E[W_5] = 5/\lambda$, [mean of a Gamma with parameters $(5, 1/\lambda)$].

   (b) $E[W_5|N(1) = 2] = 1 + 3/\lambda$ [due to the memoryless property of the exponential distribution].

   (c) $E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)] = E[N(2)] = 2\lambda$ [because the Poisson process has independent (first equality) and stationary increments (second equality)].

2. -

$$M = \sum_{i=0}^{N} I_i$$

where $\{I_i\}_{i=1,2,\ldots}$ are i.i.d. random variable, Bernoulli distributed, with parameter $p = \Pr\{X > d\} = 1 - F_X(d)$. Then

$$\begin{aligned} P_M(z) &= E\left[z^{\sum_{i=0}^{N} I_i}\right] = E\left[E\left[z^{\sum_{i=0}^{N} I_i}|N\right]\right] = \\ &= E\left[\left(E\left[z^I\right]\right)^N\right] = E\left[(zp + (1-p))^N\right] = \\ &= P_N(p(z-1)+1) = M_{p\Lambda}\left((z-1)\right) \end{aligned}$$

which is the probability generating function of a of a mixed Poisson random variable with structure r.v. $p\Lambda = (1 - F_X(d))\Lambda$.

3. We are in presence of the Polya process. $N(t)$ is a Negative Binomial with

$$\begin{aligned} p_k(t) &= \int_0^\infty \frac{\mathrm{e}^{-\lambda t}(\lambda t)^k}{k!}\frac{1}{\Gamma(r)\beta^r}\mathrm{e}^{-\lambda/\beta}\lambda^{r-1}\,d\lambda \\ &= \begin{pmatrix} r+k-1 \\ k \end{pmatrix}\left(\frac{1}{1+\beta t}\right)^r\left(\frac{\beta t}{1+\beta t}\right)^k, \end{aligned}$$

Hence $p_0(3) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$.

4. -

(a) It is $1/101 = 0.009\,901\,0$.

(b)

$$S_X(x) = \frac{100}{101}\times e^{-x} + \frac{1}{101}\times\left(\frac{2}{x+2}\right)^3,\ x>0.$$

(c) $\frac{100}{101}\times e^{-5} + \frac{1}{101}\times\left(\frac{2}{7}\right)^3 = 0.006\,902\,.$

(d)

$$\begin{aligned} \Pr\{\text{Type 2 claim } |X &> 5\} = \frac{\Pr\{X>5|\text{Type 2 claim}\}\Pr\{\text{Type 2 claim}\}}{\Pr\{X>5\}} = \\ &= \frac{\left(\frac{2}{7}\right)^3 * \frac{1}{101}}{0.006\,902\,.} = 0.033457. \end{aligned}$$

(e) Let $Z = \max\{0, X-5\}$.

$$E[Z^k] = k\int_5^\infty (x-5)^{k-1}S_X(x)dx$$

$$E[Z] = \int_5^\infty\left(\frac{100}{101}\times\exp(-x) + \frac{1}{101}\times\left(\frac{2}{x+2}\right)^3\right)dx = 0.007479.$$

$E[Z^2] = 2\int_5^\infty\left((x-5)\left(\frac{100}{101}\times\exp(-x) + \frac{1}{101}\times\left(\frac{2}{x+2}\right)^3\right)\right)dx =$
$= 0.02\,465\,8$
$P_{re} = 101\times 0.007479 + 3*(101\times 0.02\,465\,8)^{0.5} = 5.\,489\,7$

5. -

(a)

$$\begin{aligned} E[S] &= \sum_{k=1}^4 n_k q_k b_k = 2.6 \\ V[S] &= \sum_{k=1}^4 n_k q_k (1-q_k) b_k^2 = 4.318 \\ \mu_3(S) &= \sum_{k=1}^4 n_k q_k (1-q_k)(1-2q_k) b_k^3 = 7.5558 \end{aligned}$$

$$
\begin{aligned}
\mu_S &= \sum_{k=1}^{4} n_k q_k b_k = 2.6 \\
\sigma_S &= 2.077979788 \\
\gamma_S &= 0.842086096
\end{aligned}
$$

(b) Using, for instance, the NP approximation

$$\Pr\{S > 6\} = \Pr\left\{\frac{S-\mu_S}{\sigma_S} > \frac{6-2.6}{2.077979788}\right\} =$$

$$y = \frac{6-2.6}{2.077979788} = 1.636\,204\,56$$

$$z = -\frac{3}{\gamma_S} + \sqrt{\frac{9}{\gamma_S^2} + 1 + \frac{6}{\gamma_S} y} = 1.4723178$$

Hence

$$\Pr\{S > 6\} \simeq 1 - \Phi(1.4723178) = 0.07$$

(c) $\lambda = 1.7$
$f_X(1) = (0.2 + 0.6)/1.7 = 0.470\,588$
$f_X(2) = (0.1 + 0.8)/1.7 = 0.529\,412$

$$f_S(i) = \frac{\lambda}{i} \sum_{j=1}^{\min(i,2)} f_X(j) f_S(i-j)$$

$$
\begin{aligned}
f_S(0) &= 0.182683524 \\
f_S(1) &= 0.146146819 \\
f_S(2) &= 0.222873899 \\
f_S(3) &= 0.147121131 \\
\Pr\{S &> 3\} = 0.30117
\end{aligned}
$$

6. -

(a) $f_X(x) = \frac{1}{3} \times 3\exp(-3x) + \frac{2}{3}(9x\exp(-3x))$, $x > 0$, which is a mixture of an exponential (with parameter 1/3) with a Gamma (with parameters $(2, 1/3)$).
$M_X(r) = \frac{1}{3}\frac{3}{3-r} + \frac{2}{3}\left(\frac{3}{3-r}\right)^2$
$M_X(r) - 1 = \frac{5r-r^2}{(3-r)^2}$
$a_1 = \frac{5}{9}$
$[1 + 0.8a_1 r] - (M_X(r) - 1) = \frac{4r-5r^2+r^3}{(3-r)^2}$
$\int_0^\infty e^{ur}[-\psi'(u)]du = \frac{1}{1+\alpha}\frac{\alpha[M_X(r)-1]}{1+(1+\alpha)a_1 r - M_X(r)} = \frac{0.8}{1.8}\frac{5-r}{(4-r)(1-r)}$
$\psi(u) = \frac{16}{27}\exp(-u) - \frac{1}{27}\exp(-4u)$

(b) The adjustment coefficient is $R = 1$. Hence the Lundberg bound is $\psi(u) \leq \exp(-u)$.

7. -

(a) X is a Weibull with parameters $\theta = \sqrt{2}$ and $\tau = 2$. Then $VaR_{0.99}(X) = \sqrt{2}\,[-\ln 0.01]^{1/2} = 3.034\,85$.

(Or calculate the solution to $1 - e^{-\frac{1}{2}x^2} = 0.99$.)

(b) $E[X \wedge d] = \int_0^d S_X(dx) = \int_0^d e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}} \int_0^d e^{-\frac{1}{2}x^2} dx\right) = \sqrt{2\pi}\left(\Phi(d) - \frac{1}{2}\right)$.

(c) The loss elimination ratio is $\frac{E[X \wedge d]}{E[X]} = \frac{\sqrt{2\pi}\left(\Phi(d) - \frac{1}{2}\right)}{\frac{\sqrt{2\pi}}{2}} = 2\left(\Phi(d) - \frac{1}{2}\right)$.

For $d = 1:$ $\frac{E[X \wedge 1]}{E[X]} = 2\left(\Phi(1) - \frac{1}{2}\right) = 2\,(0.841345 - 0.5) = 0.682\,689$

(d) $E[X \wedge VaR_{0.99}(X)] = E[X \wedge 3.034\,85] = \sqrt{2\pi}\left(\Phi(3.034\,8) - \frac{1}{2}\right) = \sqrt{2\pi}\,(0.9988 - 0.5) = 1.250\,298$

$TVaR_{0.99}(X) = VaR_{0.99}(X) + \frac{E[X] - E[X \wedge VaR_{0.99}(X)]}{0.01} = 3.034\,85 + \frac{\frac{\sqrt{2\pi}}{2} - 1.250\,298}{0.01} = 3.336\,46$.